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# A recursive representation for runaway electrons

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**Abstract.** Among the analytic studies on runaway phenomena the most sophisticated approach to solving the Fokker–Planck equation is to divide the momentum space into five distinct regions with appropriate matching between them: various expansions of the distribution function are valid in different regions. However, the solution for two of the five expansions has not yet been completed. In this paper, a pair of recursive relations is established to solve the Fokker–Planck equation for a relativistic plasma in the runaway region. Starting from the available asymptotic solution, a representation of the distribution function may be obtained by iteration. The application of this technique to the non-relativistic problem is also treated.

## 1. Introduction

Runaway electrons are a basic ingredient of hot toroidal plasmas. Their presence has a significant effect on the tokamak operation. Understanding of the runaway behaviour is thus of practical importance. A recent review of the numerous experimental and theoretical studies was published by Knoepfel and Spong (1979) with an extensive bibliography of 189 references.

It is well known that in a plasma subject to an external force (such as an electric field), electrons with velocities higher than the critical velocity go over to a state of continuous acceleration since their dynamic friction is less than the force exerted by the electric field. This property of a plasma leads to distortions in the electron distribution function away from a Maxwellian at energies a few times thermal.

One of the earliest analytic works on the runaway problem was due to Dreicer (1959, 1960). The approach of Dreicer and the other authors cited in the references is to solve the electron Fokker–Planck equation to determine the higher-energy portion of the electron distribution function. Among these contributions, the most consistent and rigorous treatment of this problem has been given by Kruskal and Bernstein (1963). They divided the momentum space into five distinct regions with appropriate matching between them: various expansions of $f$ or $\ln f$ are valid in different regions. As a result their treatment did not contain the deficiencies of other theories, but they did not complete the solution for two of the five expansions. Although the theory is incomplete, it is significant in the sense of being the first formal approach to the problem, involving no *ad hoc* assumption about the distribution function. This approach has been extended by Cohen (1976) to the multiply ionised, multiple-species plasma, and generalised by Connor and Hastie (1975) to the relativistic plasma.

Kruskal and Bernstein (1964) also treated the simplified model of an ideal Lorentz plasma, in which electrons experience Coulomb interaction only with infinitely massive ions. They have worked out a pair of recursive relations, from which a representation of the distribution function may be obtained by iteration. In the present paper we shall generalise this technique to solve the runaway region for the general case of a relativistic plasma, and discuss its reduction to the non-relativistic problem.

## 2. General theory

Let us start with the equation derived from the suprathermal relativistic Fokker–Planck equation by Connor and Hastie (1975) for the runaway region ($q > q_c$), namely,

$$\mu \frac{\partial f}{\partial q} - \frac{\eta}{q^2} \frac{\partial}{\partial q} (\gamma^2 f) = \frac{\eta \zeta \gamma}{q^3} \frac{\partial}{\partial \mu} \left( (1-\mu^2) \frac{\partial f}{\partial \mu} \right) - \frac{1-\mu^2}{q} \frac{\partial f}{\partial \mu}, \tag{2.1}$$

where $f$ denotes the electron distribution function in zeroth approximation, $q$ is the magnitude of the normalised momentum defined as $p/(m_0 c)$, $\mu$ the cosine of the angle measured from the vector $-\boldsymbol{E}$, $\gamma$ the relativistic gamma factor, $\zeta$ the parameter defined as $\frac{1}{2}(1+Z_{\text{eff}})$, $Z_{\text{eff}}$ being the effective charge number for ions, $\eta$ the runaway parameter defined as $4\pi n_e e^3 \ln \Lambda/(m_0 c^2 E)$, and $q_c$ the critical momentum given by $q_c^2 = \eta/(1-\eta)$. Connor and Hastie (1975) have found an approximate solution for the limiting case of $q \gg 1$ and $\mu \approx 1$:

$$f = \frac{\alpha}{q\mu} \exp\left( -\frac{1+\eta}{4\eta\zeta} q \frac{1-\mu^2}{\mu} \right). \tag{2.2}$$

Integrating equation (2.1) over $\mu = -1$ to 1, we have

$$\left( \frac{\partial}{\partial q} + \frac{2}{q} \right) \int_{-1}^{1} \left( \mu - \eta \frac{\gamma^2}{q^2} \right) f \, \mathrm{d}\mu = 0,$$

which yields

$$\int_{-1}^{1} \left( \mu - \eta \frac{\gamma^2}{q^2} \right) f \, \mathrm{d}\mu = \frac{C}{q^2}. \tag{2.3}$$

Define

$$P(\mu) = \int_{-1}^{\mu} (q^2 x - \eta \gamma^2) f \, \mathrm{d}x, \tag{2.4}$$

where $P(-1) = 0$ and $P(1) = C$. If we integrate equation (2.1) with respect to $\mu$, we obtain

$$\frac{1}{q} \frac{\partial}{\partial q} P(\mu) = (1-\mu^2) \left( \frac{1}{\kappa} \frac{\partial f}{\partial \mu} - f \right), \tag{2.5}$$

where the parameter $\kappa$ is defined as $q^2/(\eta\zeta\gamma)$. Solving equation (2.5) for $f$ as a function of $\mu$, we have

$$f(\mu) = \mathrm{e}^{\kappa\mu} \left( \frac{\kappa}{q} \int_{\mu_0}^{\mu} \frac{\mathrm{d}x}{1-x^2} \mathrm{e}^{-\kappa x} \frac{\partial}{\partial q} P(x) + f_0 \, \mathrm{e}^{-\gamma/\zeta} \right), \tag{2.6}$$

where $\mu_0 = \eta(\gamma/q)^2$ is chosen as the lower limit, while $f_0$ denotes the value of $f$ at $\mu = \mu_0$. Note that $\mu = \mu_0(q)$ divides the momentum space into two separate domains: the runaway source exists only in the domain $\mu > \mu_0$.

Substituting expression (2.6) into equation (2.3), we obtain an equation for $f_0$ as follows:

$$Gf_0 + H = q^{-1}P(1), \tag{2.7}$$

where

$$G = (q/\kappa)\,\mathrm{e}^{-\gamma/\zeta}[(1-\mu_0-\kappa^{-1})\,\mathrm{e}^{\kappa} + (1+\mu_0+\kappa^{-1})\,\mathrm{e}^{-\kappa}],$$

$$H = \left(1-\mu_0-\frac{1}{\kappa}\right)\mathrm{e}^{\kappa}\int_{\mu_0}^{1}\frac{\mathrm{d}x}{1-x^2}\,\mathrm{e}^{-\kappa x}\frac{\partial}{\partial q}P(x) - \left(1+\mu_0+\frac{1}{\kappa}\right)\mathrm{e}^{-\kappa}\int_{-1}^{\mu_0}\frac{\mathrm{d}x}{1-x^2}\,\mathrm{e}^{-\kappa x}\frac{\partial}{\partial q}P(x)$$
$$-\int_{-1}^{1}\left(\mu-\mu_0-\frac{1}{\kappa}\right)\frac{\mathrm{d}\mu}{1-\mu^2}\frac{\partial}{\partial q}P(\mu).$$

We shall examine in the following whether the recursive formulae (2.6) and (2.7) may be used to attain a more accurate solution by iteration when an approximate solution such as expression (2.2) is available. For very large $q$, from the definitions

$$\kappa = q/(\eta\zeta) - \mathrm{O}(q^{-1}), \qquad \mu_0 = \eta + \mathrm{O}(q^{-2}), \tag{2.8}$$

and the $\delta$ sequence may be expressed as (Kecs and Teodorescu 1974)

$$\delta(s) = (\kappa/2\pi)^{1/2}\exp(-\tfrac{1}{2}\kappa s^2), \tag{2.9}$$

where $s$ is defined by $s^2 = 2(x-\eta)$. In this approximation, expression (2.6) may be written as

$$f(\mu) = \mathrm{e}^{\kappa(\mu-\eta)}\left[\frac{1}{1-\eta^2}\left(\frac{\pi}{2\eta\zeta q}\right)^{1/2}\frac{\partial}{\partial q}P(\eta)\lim_{s\to 0} s + f_0\right], \tag{2.10}$$

where the first term in the square bracket is very small. That is to say, in equation (2.6) the second term on the right-hand side dominates the integral term, which indicates that formula (2.6) is suitable for iteration.

## 3. Reduction to non-relativistic case

Considering multiply ionised, multiple-species plasmas, Cohen (1976) has derived an equation for the non-relativistic case valid in the runaway region as follows:

$$w(w^2\mu-1)\frac{\partial f}{\partial w} + w^2(1-\mu^2)\frac{\partial f}{\partial \mu} = \zeta\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial f}{\partial \mu}\right), \tag{3.1}$$

where $w$ denotes the electron velocity normalised with respect to the critical speed. He also found an approximate solution for the limiting case of $w \gg 1$ and $\mu \approx 1$:

$$f = \frac{\alpha}{\ln(\beta w\mu)}\exp\left(-\frac{w^2(1-\mu^2)}{4\zeta\ln(\beta w\mu)}\right). \tag{3.2}$$

If we set both $\gamma$ and $\eta$ to unity and replace $q$ by $w$, equation (2.1) is then reduced to equation (3.1) exactly. Thus the method developed in § 2 may be directly applied to the non-relativistic case. In this case, expression (2.4) becomes simply

$$P(\mu) = \int_{-1}^{\mu}(w^2x-1)f\,\mathrm{d}x, \tag{3.3}$$

while the parameters $\kappa = w^2/\zeta$ and $\mu_0 = w^{-2}$. Using equations (2.6) and (2.7), the recursive relations may be expressed as

$$f(\mu) = e^{\kappa\mu}\left(\frac{w}{\zeta}\int_{\mu_0}^{\mu}\frac{dx}{1-x^2}e^{-\kappa x}\frac{\partial}{\partial w}P(x) + f_0 e^{-1/\zeta}\right), \tag{3.4}$$

and $f_0(w)$ may be determined from

$$Gf_0 + H = P(1), \tag{3.5}$$

where

$$G = \zeta e^{-1/\zeta}\left[\left(1-\frac{1+\zeta}{w^2}\right)e^{\kappa} + \left(1+\frac{1+\zeta}{w^2}\right)e^{-\kappa}\right],$$

$$H = \left(w - \frac{1+\zeta}{w}\right)e^{\kappa}\int_{\mu_0}^{1}\frac{dx}{1-x^2}e^{-\kappa x}\frac{\partial}{\partial w}P(x) - \left(w + \frac{1+\zeta}{w}\right)e^{-\kappa}\int_{-1}^{\mu_0}\frac{dx}{1-x^2}e^{-\kappa x}\frac{\partial}{\partial w}P(x)$$
$$- \int_{-1}^{1}\left(w\mu - \frac{1+\zeta}{w}\right)\frac{\partial}{\partial w}\frac{P(\mu)}{1-\mu^2}\,d\mu.$$

## 4. Concluding remarks

In § 2, we have derived recursive formulae for a relativistic plasma in the runaway region (region V according to Connor and Hastie (1975)). A more accurate solution of the electron distribution function may be obtained by an iterative process, starting from the available asymptotic solution. The analysis for large $q$ indicates that the formulae are suitable for iteration. The reduction of these recursive relations to the non-relativistic case was discussed in § 3.

## References


Cohen R H 1976 *Phys. Fluids* **19** 239–44
Connor J H and Hastie R J 1975 *Nucl. Fusion* **15** 415–24
Dreicer H 1959 *Phys. Rev.* **115** 238–49
—— 1960 *Phys. Rev.* **117** 329–42
Gurevich A V 1961 *Sov. Phys.–JETP* **12** 904–12
Kecs W and Teodorescu P P 1974 *Applications of the Theory of Distributions in Mechanics* (Tunbridge Wells: Abacus) p 73
Knoepfel H and Spong D A 1979 *Nucl. Fusion* **19** 785–829
Kruskal M D and Bernstein I B 1963 *Princeton Plasma Lab. Report No MATT-Q-20* pp 174–81
—— 1964 *Phys. Fluids* **7** 407–18
Lebedev A N 1965 *Sov. Phys.–JETP* **21** 931–3